

# Research Statement

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My research is in Harmonic Analysis which is a rich field with applications to a variety of areas, such as: dispersive PDEs, mathematical physics, number theory, signal processing, and many more. At the start of my thesis my goal was to prove  $L^p$  estimates for Fourier integral operators with rational symbols of the type presented in section 3. After careful study of these operators, my curiosity branched out into two specific areas. The first was to find estimates for generalizations of the multilinear fractional integral operators studied by Kenig and Stein in [7], Grafakos and Kalton in [3], and also in other works such as [8], [9], and [4]. The second area of interest was to study decay estimates for multilinear oscillatory integrals, which is a broad topic initialized by Christ, Li, Tao, and Thiele in [2].

In section 1 I discuss our results for the main multilinear fractional integral operator that my advisor, Camil Muscalu, and I studied and titled “The Fractional Biest.” In section 2 I discuss results on multilinear oscillatory integrals that grew out of my participation in the MRC Harmonic Analysis conference. Lastly, in section 3, I discuss specific applications to the Fourier integral operators with rational symbol that initialized our project. Interest in the operators in section 3 stemmed from a fruitful discussion with Mihaela Ifrim and Daniel Tataru on the nonlinear Schrödinger equation during Camil Muscalu’s visit to MSRI in 2018. Each of the topics discussed have strong potential for ongoing work in which techniques from partial differential equations, more general time-frequency analysis, and other areas of analysis may be applicable.

## 1 The Fractional Biest Operator

A major portion of my thesis work was dedicated to proving  $L^{p_1} \times L^{p_2} \times L^{p_3} \rightarrow L^r$  estimates for the fractional Biest operator

$$I_{\alpha,\beta}(f, g, h)(x) := \int_{\mathbb{R}^2} f(x-t)g(x+s+t)h(x-s)K_{\alpha,\beta}(t, s) dsdt \quad (1.1)$$

where  $K_{\alpha,\beta}(t, s) = \frac{1}{|t|^\alpha |s|^\beta}$  and  $\alpha, \beta \in (0, 1)$ . Note that if we replace  $K_{\alpha,\beta}(t, s)$  with the kernel  $K(t, s) = \frac{1}{t \cdot s}$ , then the operator in (1.1) becomes the known Biest operator studied by Camil Muscalu, Terence Tao, and Christoph Thiele in [12] for which the authors prove  $L^{p_1} \times L^{p_2} \times L^{p_3} \rightarrow L^r$  bounds in the range  $1 < p_1, p_2, p_3 \leq \infty$  and  $2/5 < r < \infty$ , for exponents satisfying

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r}.$$

### 1.1 $L^p$ Estimates for the Fractional Biest

Our main result regarding the fractional Biest operator (1.1) is as follows:

**Theorem 1.1.** *Assume that  $0 < \alpha, \beta < 1$ ,  $1 \leq p_1, p_2, p_3 \leq \infty$ ,  $0 < r < \infty$ , and*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \alpha + \beta = \frac{1}{r} + 2. \quad (1.2)$$

*Then, we have:*

*i) If  $1 < p_i, i = 1, 2, 3$ ,*

$$\|I_{\alpha,\beta}(f, g, h)\|_{L^r(\mathbb{R})} \lesssim \|f\|_{L^{p_1}(\mathbb{R})} \|g\|_{L^{p_2}(\mathbb{R})} \|h\|_{L^{p_3}(\mathbb{R})}.$$

*ii)  $1 \leq p_i, i = 1, 2, 3$ , and at least one of the  $p_i$  is one, then instead we obtain the analogous restricted weak type estimate. That is*

$$\|I_{\alpha,\beta}(f, g, h)\|_{L^{r,\infty}(\mathbb{R})} \lesssim |E_1|^{1/p_1} |E_2|^{1/p_2} |E_3|^{1/p_3}$$

*for  $|f| \leq \chi_{E_1}, |g| \leq \chi_{E_2}, |h| \leq \chi_{E_3}$  where  $E_i$  are sets of finite measure.*

The exponent relation (1.2) is dictated by the scaling properties of  $I_{\alpha,\beta}$ . In Theorem 1.1 we obtain the full expected range of strong type  $L^p$  estimates, with the exception of uncertainty at the endpoints, i.e. exactly when  $p_i = 1$  for any  $i$ . The Biest operator mentioned above can be seen as the limiting case of the fractional Biest operator as  $\alpha, \beta$  approach one. Notice that our range of estimates in the limiting case is consist with the range obtain in [12], however for the fractional Biest, as  $\alpha, \beta$  approach one, our estimates hold arbitrarily close to the end point  $r = 1/3$  whereas this sharp range is not known yet for the Biest operator.

Our methods of proving Theorem 1.1 consist of proving restricted weak type estimates in the full range including the end point  $p_1 = p_2 = p_3 = 1$ . We then convert these restricted weak type estimates into the strong type estimates stated in (i) of Theorem 1.1 by applying the multilinear interpolation theorems used in [12].

In order to obtain restricted weak type estimates, we use a duality lemma similar to that of Ch. 2 in [13]. However, in our case we dualize through  $L^{1/3}$  which allows us to reach the borderline estimate uniformly in  $\alpha, \beta$ , this is a technique that was used by Cristina Benea and Camil Muscalu in [1]. After the careful dualization and discretization process, for  $|f| \leq \chi_{E_1}, |g| \leq \chi_{E_2}, |h| \leq \chi_{E_3}$ , and  $|F| \leq \chi_E$ , we obtain the model:

$$\begin{aligned}
& \|I_{\alpha,\beta}(f,g,h) \cdot F\|_{L^{1/3}}^{1/3} \lesssim \\
& \sum_{J \text{ dyadic}} |J|^{\frac{2-\beta}{3}} \left( \sum_{\substack{I \subseteq J \\ I \text{ dyadic}}} |I|^{\frac{3-\alpha}{2}} \text{avg}_I(f)^{1/2} \widetilde{\text{avg}}_{I,J}(g,h)^{1/2} \text{avg}_I(F)^{1/2} \right)^2 \text{avg}_J(F)^{1/3}.
\end{aligned} \tag{1.3}$$

We can assume with out loss of generality that all functions are non-negative since  $I_{\alpha,\beta}$  is a positive operator. For a given function  $H$ ,  $\text{avg}_I(H) := \frac{1}{|I|} \int_I H(x) dx$ , and

$$\widetilde{\text{avg}}_{I,J}(g,h) := \int_{R(I,|J|)} g(\tilde{s})h(\tilde{t}) d\tilde{s}d\tilde{t}$$

where, as shown in Figure 1,  $R(I, |J|)$  is a rectangle index by  $(I, |J|)$ . The rectangle  $R(I, |J|)$  was obtained from a careful change of variables and a translation that depends precisely on interval  $I$ . This change of variables was a natural choice as it allowed us to separate  $f$  as an average on the small scales only, while the newly defined average encoded information of both  $I$  and  $J$ , and hence allowed us to carefully tackle the presence of two scales in the decomposition.

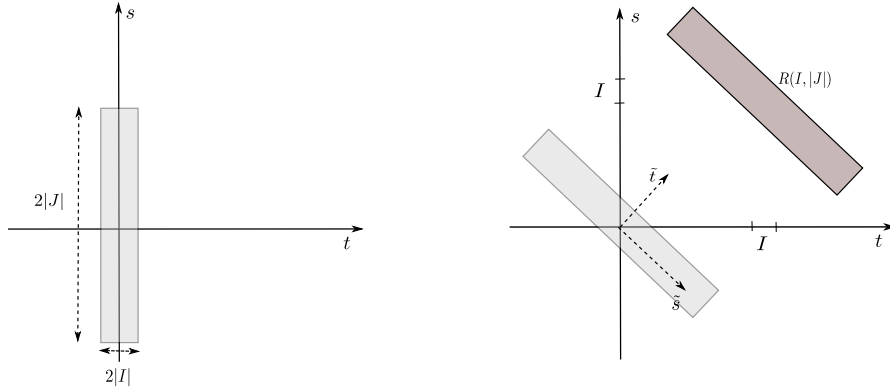


Figure 1: The left side shows the rectangles originally faced and the right side shows rectangles obtained after a change of variables and translation.

An important fact that we prove about  $\widetilde{\text{avg}}_{I,J}(g,h)$  is that it satisfies the main properties needed to run a successful stopping time argument. That is, it satisfies simplified versions of the intricate “size” and “energy” type estimates described in Ch. 6 of [13].

While our work is the first to our knowledge to study the fractional Biest operator, a similar fractional analogue of the bilinear Hilbert transform (BHT) was studied by Kenig

and Stein in [7] and independently by Grafakos and Kalton in [3]. Namely, they showed that

$$I_\alpha(f, g)(x) := \int_{\mathbb{R}} \frac{f(x-t)g(x+t)}{|t|^\alpha} dt \quad (1.4)$$

satisfies estimates

$$\|I_\alpha(f, g)\|_{L^r} \lesssim \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}$$

for  $\frac{1}{p_1} + \frac{1}{p_2} + \alpha = \frac{1}{r} + 1$ ,  $1 < p_1, p_2 \leq \infty$ ,  $r > 0$ , and if either  $p_1, p_2$  are one, then they showed that the same estimate holds by replacing  $L^r$  with  $L^{r, \infty}$ . Similar to our previous observation, in the limit as  $\alpha$  approaches one estimates for  $I_\alpha$  approach those satisfied by the BHT, but for the fractional BHT the borderline estimates below  $r = 2/3$  are reached whereas estimates for  $1/2 < r < 2/3$  are not known for the BHT. More information on the BHT operator itself, can be found but is not limited to the work in [5], [10], [15], [16], and [17].

The method of proof used in [7] and [3] takes advantage of the scaling of  $I_\alpha$  and the most natural extension of their argument does not apply to operators such as (1.1) whose discretized model is comprised of mixed scale averages. Using our method, we are not only able to prove estimates for the fractional Biest, but we also recover the strong  $L^p$  type estimates in [7] and [4], and in ongoing work we plan to generalize our method to more advanced multilinear fractional operators, such as multiple multilinear iterations of the aforementioned ones.

## 2 Oscillatory Integral Operators

In June 2018, I was a participant of the MRC Harmonic Analysis conference on new developments on oscillatory integrals. Through this conference I met my collaborators Zhen Zeng and Kevin O'Neill, and we were granted the MRC collaboration Research Travel Grant which allowed us to meet at the University of Pennsylvania to work on a project proposed by Philip Gressman.

Our project was motivated by the work of Christ, Li, Tao, and Thiele in [2] in which they initialize the systematic study of general multilinear oscillatory integrals. Namely, those of the form

$$T(f_1, \dots, f_n) = \int_{\mathbb{R}^m} e^{i\lambda S(x)} \prod_{j=1}^n f_j(\pi_j(x)) \phi(x) dx, \quad (2.1)$$

where  $\lambda \in \mathbb{R}$  is a parameter,  $S : \mathbb{R}^m \rightarrow \mathbb{R}$  is a real-valued measurable function,  $\phi \in C_0(\mathbb{R}^m)$  is a cutoff function containing the origin in its support, and the  $\pi_j$  are orthogonal

projections onto proper subspaces  $V_j$  of  $\mathbb{R}^m$  of common dimension satisfying some additional properties that we will forgo mentioning here. Our goal was to determine decay rates of these oscillatory integral operators.

## 2.1 Decay of Multi-linear Oscillatory Integral Operators in $\mathbb{R}^2$

This project led to the publication of [14] in which we prove Theorem 2.1 below. The details of the hypothesis are included for completion, but we will not go into details of their meaning in this exposition.

**Theorem 2.1.** *Let  $n \geq 4$  and let  $a_j = (b_j, c_j) \in \mathbb{R}^2 \setminus \{0\}$  ( $1 \leq j \leq n$ ) lie in general position and define  $D_n = \prod_{j=1}^n (c_j \partial_x - b_j \partial_y)$ . Then, there exists  $C > 0$ , depending only on  $\phi$  and the  $a_j$ , such that if  $|D_n S(x, y)| \geq 1$  for all  $(x, y)$  in the convex hull of the support of  $\phi$ ,*

$$\left| \int_{\mathbb{R}^2} e^{i\lambda S(x, y)} \prod_{j=1}^n f_j(a_j \cdot (x, y)) \phi(x, y) dx dy \right| \leq C |\lambda|^{-2^{1-n}} \prod_{j=1}^n \|f_j\|_{p_j} \quad (2.2)$$

for

$$\mathbf{p} = \left( 2, 2, \frac{2^{n-1}}{2^{n-2}-1}, \dots, \frac{2^{n-1}}{7}, \frac{2^{n-1}}{3} + \epsilon, 2^{n-1} - \frac{9 \cdot 2^{n-3}\epsilon}{2^{n-3} + 3\epsilon} \right) \quad (2.3)$$

and  $0 < \epsilon < 1$ .

The theorem above is a generalization of a result of Gressman and Xiao in [6] where they proved that

$$\left| \iint e^{i\lambda S(x, y)} f(x) g(y) h(x+y) \phi(x, y) dx dy \right| \lesssim |\lambda|^{-1/4} \|f\|_p \|g\|_q \|h\|_r \quad (2.4)$$

if  $|\partial_x \partial_y (\partial_x - \partial_y) S(x, y)| \geq 1$  and whenever  $p, q, r \in [2, 4)$  and  $p^{-1} + q^{-1} + r^{-1} = 5/4$ .

In the proof of Theorem 2.1 we used the trilinear oscillatory integral decay estimate of Gressman and Xiao mentioned above and proceed by induction. In the induction step, a  $TT^*$  argument is used to reduce the degree of linearity by 1.

In an ongoing work with Kevin O'Neill and Zhen Zeng, we are making progress in studying trilinear oscillatory integrals similar to (2.1) but in higher dimensions. Namely, our goal is to generalize an oscillatory integral result found in [11].

## 3 Applications to Multilinear Fourier Integral Operators with Rational Symbol

My interest in the content of sections 1 and 2 originated from a more general part of my thesis work, which was to make sense of and find estimates for Fourier integral operators

with rational symbol of the form

$$T_P(f_1, f_2, \dots, f_m)(x) := \int_{\mathbb{R}^{nm}} \frac{1}{P(\xi_1, \xi_2, \dots, \xi_m)} \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \dots \hat{f}_m(\xi_m) e^{2\pi i(x \cdot (\xi_1 + \xi_2 + \dots + \xi_m))} d\xi_1 \dots d\xi_m, \quad (3.1)$$

where  $\hat{f}_i \in \mathcal{S}(\mathbb{R}^n)$ ,  $x \in \mathbb{R}^n$ ,  $\xi_i := (\xi_{i,1}, \xi_{i,2}, \dots, \xi_{i,n}) \in \mathbb{R}^n$ , and  $P$  is a polynomial in  $nm$  variables. Note that if  $P \equiv 1$  then  $T(f_1, \dots, f_m)(x) = f_1 \cdot f_2 \dots f_m(x)$ . If  $P$  is degree one, then  $T_P$  can be written in terms of translations, and estimates can be obtained using Hölder inequalities. In the cases we study,  $P$  is more involved and requires us to carefully define  $T_P$  in the principal value sense.

After very careful manipulations, for certain  $P$ , we manage to write  $T_P$  in terms of products of solutions to various dispersive PDEs, like the Schrödinger or KdV equations. The easiest such example is:

$$\begin{aligned} T_{P_1}(f_1, f_2)(x_1, x_2) &= \int_{\mathbb{R}^4} \frac{1}{\xi_{1,1} + \xi_{2,2}^2} \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) e^{2\pi i(x_1, x_2) \cdot (\xi_1 + \xi_2)} d\xi_1 d\xi_2 \\ &= \int_{\mathbb{R}} \text{sign}(t) f_1(x_1 + t, x_2) S_2^t[f_2](x_1, x_2) dt \end{aligned} \quad (3.2)$$

where  $S_2^t[f_2](x_1, x_2)$  represents the solution to the Schrödinger equation with initial condition  $f_2$  in just the second coordinate. In this case, one can proceed to apply Strichartz estimates to obtain:

$$\|T_{P_1}(f_1, f_2)\|_{L_{x_1}^{r_1} L_{x_2}^{r_2}} \lesssim \|f_1\|_{L_{x_1}^{p_1} L_{x_2}^{p_2}} \|f_2\|_{L_{x_1}^{r_1} L_{x_2}^{r_2}} \quad (3.3)$$

with  $(\frac{1}{2} + \frac{1}{p_2} - \frac{1}{r_2}) + \frac{2}{p_1} = 2$ ,  $1 \leq p_1 \leq \frac{4}{3}$ ,  $0 \leq \frac{1}{r_2} - \frac{1}{p_2} \leq \frac{1}{2}$ . In other cases we formulate  $T_P$  in terms of more general oscillatory integral operators and apply Hörmander's oscillatory integral theorem. Omitting the intermediate steps, the easiest non-trivial example is

$$\begin{aligned} |T_{P_2}(f_1, f_2, f_3)(x_1, x_2, x_3)| &= \left| \int_{\mathbb{R}^9} \frac{\hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \hat{f}_3(\xi_3)}{\xi_{1,1} \xi_{2,2} + \xi_{3,3}} e^{2\pi i(x_1, x_2, x_3) \cdot (\xi_1 + \xi_2 + \xi_3)} d\xi_1 d\xi_2 d\xi_3 \right| \\ &\lesssim \int_{\mathbb{R}} \frac{|h(x_1, x_2, x_3 + t)|}{|t|^{1/2}} \|f(\cdot, x_2, x_3)\|_{L_{x_1}^2} \|g(x_1, \cdot, x_3)\|_{L_{x_2}^2} dt. \end{aligned} \quad (3.4)$$

Applying the classic Hardy-Littlewood-Sobolev fractional integration theorem coupled with Hölder, we obtain the following mixed norm estimates:

$$\|T_{P_2}(f_1, f_2, f_3)\|_{L_{x_1}^{r_1} L_{x_2}^{r_2} L_{x_3}^{r_3}} \lesssim \|f_1\|_{L_{x_2}^{p_2} L_{x_3}^{p_3} L_{x_1}^2} \|f_2\|_{L_{x_1}^{q_1} L_{x_3}^{q_3} L_{x_2}^2} \|f_3\|_{L_{x_1}^{u_1} L_{x_2}^{u_2} L_{x_3}^{u_3}} \quad (3.5)$$

with  $\frac{1}{p_i} + \frac{1}{u_i} = \frac{1}{r_i}, i = 1, 2$  and  $\frac{1}{p_3} + \frac{1}{q_3} + \frac{1}{u_3} = \frac{1}{r_3} + \frac{1}{2}, u_3 > 2$ .

The examples above laid the groundwork for us to explore operators of type  $T_P$  with more complicated choices for  $P$ . Having at hand a wide variety of oscillatory integral decay estimates together with appropriate multilinear fractional integration theorems appears to be a necessity for us as we begin to develop a systematic understanding of the operators  $T_P$ . To conclude, below are two theorems that resulted from our studies.

**Theorem 3.1.** *For  $x = (x_1, x_2, x_3) \in \mathbb{R}^3$  and  $f_i \in \mathcal{S}(\mathbb{R}^3)$ , consider*

$$T_{P_3}(f_1, f_2, f_3, f_4)(x) := \int_{\mathbb{R}^{12}} \frac{\hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \hat{f}_3(\xi_3) \hat{f}_4(\xi_4)}{\xi_{1,1} \xi_{2,2} + (\xi_{3,3} - \xi_{4,3})} e^{2\pi i x \cdot (\xi_1 + \xi_2 + \xi_3 + \xi_4)} d\xi_1 d\xi_2 d\xi_3 d\xi_4.$$

*Then,  $|T_{P_3}(f_1, f_2, f_3, f_4)|$  can be bounded in terms of the fractional BHT and*

$$\|T_{P_3}(f_1, f_2, f_3, f_4)\|_{L_{x_1}^{r_1} L_{x_2}^{r_2} L_{x_3}^{r_3}} \lesssim \|f_1\|_{L_{x_2}^{p_2} L_{x_3}^{p_3} L_{x_1}^2} \|f_2\|_{L_{x_1}^{q_1} L_{x_3}^{q_3} L_{x_2}^2} \|f_3\|_{L_{x_1}^{u_1} L_{x_2}^{u_2} L_{x_3}^{u_3}} \|f_4\|_{L_{x_1}^{v_1} L_{x_2}^{v_2} L_{x_3}^{v_3}}$$

*with  $\frac{1}{p_i} + \frac{1}{q_i} + \frac{1}{u_i} + \frac{1}{v_i} = \frac{1}{r_i} + \frac{1}{2}$  for  $i = 1, 2, 3$  and  $p_1 = q_2 = 2$ .*

**Theorem 3.2.** *For  $x \in \mathbb{R}^3$  and  $f_i \in \mathcal{S}(\mathbb{R}^3)$ , consider*

$$T_{P_4}(f_1, \dots, f_7)(x) := \int_{\mathbb{R}^{21}} \frac{\hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \hat{f}_3(\xi_3) \dots \hat{f}_7(\xi_7)}{(\xi_{1,1} \xi_{2,2} + \xi_{5,3} - \xi_{6,3})(\xi_{3,3} \xi_{4,3} + \xi_{5,3} - \xi_{7,3})} e^{2\pi i x \cdot (\xi_1 + \dots + \xi_7)} d\xi_1 \dots d\xi_7. \quad (3.6)$$

*Then,  $|T_{P_4}(f_1, \dots, f_7)|$  can be bounded in terms of the fractional Biest and*

$$\|T_{P_4}(f_1, \dots, f_7)\|_{L_{x_1}^{r_1} L_{x_2}^{r_2} L_{x_3}^{r_3}} \lesssim \|f_1\|_{L_{x_2}^{p_2} L_{x_3}^{p_3} L_{x_1}^2} \|f_2\|_{L_{x_1}^{q_1} L_{x_3}^{q_3} L_{x_2}^2} \|f_3\|_{L_{x_1}^{u_1} L_{x_2}^{u_2} L_{x_3}^2} \|f_4\|_{L_{x_1}^{v_1} L_{x_2}^{v_2} L_{x_3}^2} \quad (3.7)$$

$$\|f_5\|_{L_{x_1}^{\tilde{p}_1} L_{x_2}^{\tilde{p}_2} L_{x_3}^2} \|f_6\|_{L_{x_2}^{\tilde{q}_2} L_{x_3}^{\tilde{q}_3} L_{x_1}^2} \|f_7\|_{L_{x_2}^{\tilde{u}_2} L_{x_3}^{\tilde{u}_3} L_{x_1}^2} \quad (3.8)$$

*with  $\frac{1}{p_i} + \frac{1}{q_i} + \frac{1}{u_i} + \frac{1}{v_i} + \frac{1}{\tilde{p}_i} + \frac{1}{\tilde{q}_i} + \frac{1}{\tilde{u}_i} = \frac{1}{r_i} + \frac{1}{2}, i = 1, 2$  with  $p_1 = q_2 = 2$ , and  $\frac{1}{p_3} + \frac{1}{q_3} + \frac{1}{\tilde{p}_3} + \frac{1}{\tilde{q}_3} + \frac{1}{\tilde{u}_3} = \frac{1}{r_3} + 1$*

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